sl(N) Onsager's Algebra and Integrability

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We define an $sl(N)$ analog of Onsager's algebra through a finite set of relations that generalize the Dolan–Grady defining relations for the original Onsager's algebra. This infinite-dimensional Lie algebra is shown to be isomorphic to a fixed-point subalgebra of $sl(N)$ loop algebra with respect to a certain involution. As the consequence of the generalized Dolan–Grady relations a Hamiltonian linear in the generators of $sl(N)$ Onsager's algebra is shown to possess an infinite number of mutually commuting integrals of motion.

1. INTRODUCTION

The seminal work of Onsager of 1944 on the exact solution of the planar Ising model[1] has been a source of a considerable part of the subsequent developments in the field of exactly solvable models in statistical mechanics and field theory. One branch of these developments, which originated with the star–triangle relation[1,2] and let to the Yang–Baxter equation and later to theory of quantum groups, was particularly vigorous.

Yet the star–triangle relation did not play any essential role in Onsager's original solution of the 2D Ising model. Indeed it is only mentioned in ref. 1. The crucial part was played by a certain infinite-dimensional Lie algebra which is now called Onsager's algebra and by what one may call associated representation theory.

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This algebra can be described by introducing the basis $\{A_m, G_n\}$, $m = 0, \pm 1, \pm 2,...; n = 1, 2,....$ Commutation relations in this basis are

$$[A_l, A_m] = 4G_{l-m}, \qquad l \geqslant m \tag{1}$$

$$[G_l, A_m] = 2A_{m+l} - 2A_{m-l} \tag{2}$$

$$[G_l, G_m] = 0 \tag{3}$$

Even though this algebra was at the center of Onsager's original solution of the 2D Ising model, it received substantially less attention in subsequent years than the star–triangle relation. In the context of the Ising model the algebraic method of Onsager was superseded by simpler and more powerful methods that rely on the equivalence of the 2D Ising model to a free-fermionic theory[3] and the dimer problem.[4] This caused Onsager's algebra to remain in a shadow for quite a long time. The situation changed in the 1980s when several important advances related to Onsager's algebra took place. We briefly review some of these below in order to set up a background for the subsequent discussion.

Dolan and Grady[5] considered Hamiltonians H of the form (in a different notation)

$$H = A_0 + kA_1 \tag{4}$$

where k is a constant and A_0, A_1 are operators. They showed that the following pair of conditions imposed upon the operaors A_0 and A_1 (Dolan–Grady relations)

$$[A_0, [A_0, [A_0, A_1]]] = 16[A_0, A_1] \tag{5}$$

$$[A_1, [A_1, [A_1, A_0]]] = 16[A_1, A_0] \tag{6}$$

are sufficient to guarantee that H belongs to an infinite family of mutually commuting operators, integrals of motion for H. To be more precise, only one of the relations (5), (6) was considered in ref. 5: the second one was produced as a consequence of certain duality condition imposed on the operators A_0, A_1. The Dolan–Grady relations in the form (5), (6)—without the assumption of duality—were first discussed in refs. 8 and 9. Based on the work of Dolan and Grady, Perk[7] and Davies[8] established, moreover, that the Lie algebra generated by two "letters" A_0, A_1 subject to the relations (5), (6) is precisely Onsager's algebra as defined in (1)–(3). The elements A_0, A_1 of the basis $\{A_m, G_n\}$ are identified with A_0, A_1 in (5), (6) and all the rest are expressed as commutators of A_0 and A_1.

The representation of the generators A_0 and A_1 which was considered by Onsager is

$$A_0 = \sum_{i=1}^{L} \sigma_i^x, \qquad A_1 = \sum_{i=1}^{L} \sigma_i^z \sigma_{i+1}^z \tag{7}$$

and after substitution in (4) gives the Hamiltonian of the transverse Ising chain. This Hamiltonian is defined on a periodic chain of length L; and $\sigma_i^{x,z}$ are Pauli matrices representing local spins on a site with a number i. It is well known that the Jordan–Wigner transformation brings this Hamiltonian to a free-fermionic form.[3]

The important question of whether there are other representations of the relations (5), (6) leading to nearest neighbor spin Hamiltonians that are not free was answered affirmatively by von Gehlen and Rittenberg in 1985,[6] who found a family of such representations. For every integer $M \geqslant 2$ they define

$$A_0 = \frac{4}{M} \sum_{i=1}^{L} \sum_{n=1}^{M-1} \frac{X_i^n}{1 - \omega^{-n}}, \qquad A_1 = \frac{4}{M} \sum_{i=1}^{L} \sum_{n=1}^{M-1} \frac{Z_i^n Z_{i+1}^{M-n}}{1 - \omega^{-n}} \tag{8}$$

where X_i, Z_i are local Z_M spin operators satisfying $[X_i, X_j] = [Z_i, Z_j] = 0$, $Z_i X_j = \omega^{\delta_{ij}} X_j Z_i$, $Z_i^M = X_i^M = I$; and $\omega = \exp(2\pi i/M)$. When $M = 2$ this representation coincides with Onsager's original representation (7) for the Ising model. For arbitrary integer M the spin-chain Hamiltonians of the form (1) with A_0, A_1 given by (8) were later shown to be certain—so-called *superintegrable*[14,12,17]—specializations of the spin chains generated by the 2D chiral Potts model.[10–14] Von Gehlen and Rittenberg also observed numerically certain Ising-like structure of some eigenvalues in the spectrum of these Hamiltonians. In ref. 14 and 15 the $M = 3$ case was solved analytically and this structure was shown to hold for all eigenvalues. This Ising-like form of eigenvalues was later rigorously established by Davies[9] to be a consequence of Onsager's algebra; Davies proved that all eigenvalues of a Hamiltonian of the form (4) defined in a finite-dimensional Hilbert space fall into multiplets parametrized by two real numbers α, β, positive integer n, n real numbers θ_i, and n nonnegative integers s_i; eigenvalues which belong to such a multiplet are given by the formula

$$\alpha + k\beta + \sum_{j=1}^{n} 4m_j (1 + k^2 + 2k\cos\theta_j)^{1/2}, \qquad m_j = -s_j, -s_j+1,..., s_j \tag{9}$$

Classification of the finite-dimensional representations of Onsager's algebra which leads to this form of the eigenvalues was carried out by Davies[9] and subsequently by Roan.[10]

Onsager's algebra by itself does not define the parameters $\{\alpha, \beta, \theta_i, s_i\}$ entering into the eigenvalue formula (9). To find these for a given representation of the generators A_0, A_1 in some Hilbert Space $\mathscr{H}$, one needs to find the decomposition of $\mathscr{H}$ into irreducible subrepresentations of Onsager's algebra.[9,17]

For the superintegrable chiral Potts Hamiltonians given by (8) the complete spectrum of eigenvalues has been found in ref. 15 for the three state case $M = 3$ with the aid of a certain cubic relation satisfied by the transfer matrix of the chiral Potts model. For general M the eigenvalues of the ground-state sector were obtained in ref. 12 by use of an inversion identity for the same transfer matrix; in ref. 16 this result was extended to other eigenvalues. Further results related to the superintegrable chiral Potts model can be found in ref. 18.

Now we come to the motivation and the subject of the present paper. Since the work of Onsager it has been known that there exists an intimate relationship between Onsager's algebra and $sl(2)$. This relationship was clarified by Roan,[10] who built on the earlier work of Davies.[8,9] Roan showed that Onsager's algebra given by (5), (6) [or equivalently by (1)–(3)] is isomorphic to the fixed-point subalgebra of $sl(2)$ loop algebra $\mathscr{L}(sl(2))$ (or alternatively of its central extension $A_1^{(1)}$; ref. 19) with respect to the action of a certain involution. Indeed one can easily guess that there should be a connection between Onsager's algebra and the Kac–Moody algebra $A_1^{(1)}$ looking at the Dolan–Grady relations whose left-hand sides coincide with the left-hand sides of the Serre relations for $A_1^{(1)}$.[19]

This obviously raises the question of whether one can find a generalization of Onsager's algebra related in some way to $\mathscr{L}(sl(N))$ for $N \geqslant 3$ and to other loop or Kac–Moody algebras. In particular, can one find some generalizations of the Dolan–Grady conditions (5), (6) leading to integrability in the sense of existence of infinite series of integrals of motion in involution?

The aim of this paper is to propose such a generalization related to $sl(N)$ loop algebra for $N \geqslant 3$. We reserve the discussion of the generalizations related to loop algebras over other classical Lie algebras for a future publication. In order to avoid possible confusion we note that N here stands for the rank of the Lie algebra and not for the number of states in the chiral Potts model, which was denoted by M above. Let us summarize the results. We consider Hamiltonians of the form

$$H = k_1 e_1 + k_2 e_2 + \cdots + k_N e_N, \qquad N \geqslant 3 \tag{10}$$

where k_i are some arbitrary numerical constants and e_i are linear operators.

We find that if the operators e_i satisfy certain commutation relations [generalized Dolan–Grady relations given by the formulas (11)–(12) below], then the Hamiltonian H is a member of an infinite family of mutually commuting integrals of motion [see formula (48)]. Each of these integrals is linear in the coupling constants k_i and is explicitly expressed in terms of e_i. This property follows from the fact that the Lie algebra generated by e_i subject to the relations (11), (12) [we call it the $sl(N)$ Onsager's algebra and denote it by $\mathscr{A}_N$] is isomorphic to the fixed-point subalgebra of $sl(N)$ loop algebra under the action of a certain involution.

The important problem is to find examples of interesting Hamiltonians of the form (10). The only example of a Hamiltonian satisfying the generalized Dolan–Grady relations that we have been able to find so far is the Hamiltonian of an inhomogeneous periodic Ising chain of length L. To write down such a Hamiltonian we first define a sequence of operators

$$g_{2k-1} = \tfrac{1}{2}\sigma_k^x, \qquad g_{2k} = \tfrac{1}{2}\sigma_k^z \sigma_{k+1}^z, \qquad k = 1, 2,..., L$$

Then we take the operators e_i entering into the Hamiltonian to be given by $e_i = g_i$ for $i = 1, 2,..., 2L$. These operators satisfy (11), (12) for the $sl(2L)$ Onsager's algebra. After substitution of these operators into (10) we get the completely inhomogeneous transverse Ising chain with inhomogeneities k_i. If we take $2L = mN$ for some integer m, then defining the operators e_i by $e_i = \sum_{s=0}^{m-1} g_{Ns+i}$ for $i = 1, 2,..., N$, we get a representation of $sl(N)$ Onsager's algebra. The Hamiltonian corresponding to this representation is the Hamiltonian of the transverse Ising chain with periodic inhomogeneities. These Hamiltonians have been known for a long time to be free-fermionic.[20,21] So the outstanding unsolved problem is to find representations of $sl(N)$ Onsager algebra that give rise to models that cannot be mapped onto free fermions, or to prove that such representations do not exist.

Now we outline the contents of the paper. In Section 2 we give the definition of the main object of our study: $sl(N)$ Onsager's algebra, which we denote by $\mathscr{A}_N$. We specify the algebra by giving N generators and a finite number of defining relations that generalize the Dolan–Grady conditions. In Section 3 we discuss an involution of the $sl(N)$ loop algebra and its fixed-point subalgebra $\tilde{\mathscr{A}}_N$; as we shall see later, this subalgebra is isomorphic to $\mathscr{A}_N$. In order to prepare the proof of this isomorphism in subsequent sections we introduce two convenient bases in $\tilde{\mathscr{A}}_N$. In Section 4 we study the structure of the algebra $\mathscr{A}_N$ and prove a proposition which gives a set of elements that span $\mathscr{A}_N$ as a linear space. We achieve this result by computing commutators of the generators with the elements of this set. In Section 5 the isomorphism of the Lie algebras $\mathscr{A}_N$ and $\tilde{\mathscr{A}}_N$ is established. Due to the results established in Section 3, this gives us a basis

of the Algebra $\mathscr{A}_N$ and commutation relations among the elements of this basis. In Section 6 the knowledge of the basis and commutation relations enables us to find an infinite family of mutually commuting elements in the Lie algebra $\mathscr{A}_N$. The Hamiltonian (10) is one of these integrals of motion.

2. DEFINITION OF $\mathscr{A}_N$—$sl(N)$ ANALOG OF ONSAGER'S ALGEBRA

In this section we introduce the Lie algebra $\mathscr{A}_N$ which is a generalization of the original Onsager's algebra to the $sl(N)$, $N \geqslant 3$ case. The relation of this algebra to $sl(N)$ or, more precisely, to the loop algebra of Laurent polynomials with values in $sl(N)$ will be explained later in Section 5. In order to define the algebra $\mathscr{A}_N$ it is convenient to consider the Dynkin graph of the type $A_{N-1}^{(1)}$:
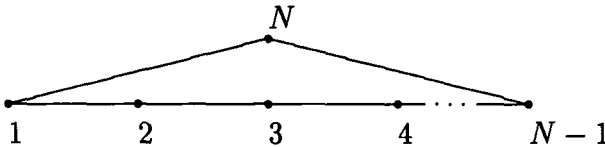
We label the vertices of this graph by i ranging from 1 to N, $i + N \overset{\text{def}}{=} i$. To a vertex with a label i we attach a letter $\mathbf{e}_i$. Then we define $\mathscr{A}_N$ to be a complex Lie algebra generated by the letter $\mathbf{e}_i$, $i = 1, 2,..., N$, subject to the following defining relations:

$$[\mathbf{e}_i, [\mathbf{e}_i, \mathbf{e}_j]] = \mathbf{e}_j \qquad \text{if } i \text{ and } j \text{ are adjacent vertices} \qquad (11)$$

$$[\mathbf{e}_i, \mathbf{e}_j] = 0 \qquad \text{if } i \text{ and } j \text{ are not adjacent} \qquad (12)$$

As a linear space the algebra $\mathscr{A}_N$ is a linear span of all multiple commutators of $\mathbf{e}_i$ between themselves taken modulo the relations (11), (12).

Let us now introduce some notations. We shall often be working with multiple commutators nested to the right, that is, expressions of the form $[a_{k_1}, [a_{k_2}, [a_{k_3},..., [a_{k_{m-1}}, a_{k_m}] \cdots]]]$, where a_{k_i} are some elements in the Lie algebra $\mathscr{A}_N$. For such a commutator we shall use the notation

$$[a_{k_1}, [a_{k_2}, [a_{k_3},..., [a_{k_{m-1}}, a_{k_m}] \cdots]]] \overset{\text{def}}{=} [a_{k_1}, a_{k_2}, a_{k_3},..., a_{k_{m-1}}, a_{k_m}]$$

In multiple commutators in which the generators $\mathbf{e}_i$ appear, we shall replace the symbol $\mathbf{e}_i$ by i, for example,

$$[\mathbf{e}_5, [\mathbf{e}_1, [\mathbf{e}_2, \mathbf{e}_5]]] \overset{\text{def}}{=} [5, 1, 2, 5] \overset{\text{def}}{=} [5, [1, 2, 5]] \overset{\text{def}}{=} \text{etc.}$$

Now we define elements of $\mathscr{A}_N$ which will play an important role in the subsequent discussion. These elements are denoted by $S_k(r)$ and are defined as follows:

$$S_k(r) \overset{\text{def}}{=} [k, k+1, k+2,..., k+r-1]; \qquad k = 1, 2,..., N, \qquad r = 1, 2, 3,...$$

(13)

We shall call such an element *a string* of length r. Strings are cyclic in their subindices: $S_{k+N}(r) = S_k(r)$. Strings of length 1 are the generators of $\mathscr{A}_N$: $S_k(1) = \mathbf{e}_k$. The string of length 0 is by convention equal to zero. As we will see in Section 4, the whole algebra $\mathscr{A}_N$ is spanned by strings as a linear space. Strings are linearly independent except that the sum of all closed strings (i.e., strings whose length is divisible by N) of a given length vanishes. This will be established in Section 5.

The algebra $\mathscr{A}_N$ has an automorphism of order N which we will use later. This automorphism, which we denote C, is defined by cyclic permutation of the generators:

$$C: \mathbf{e}_i \to \mathbf{e}_{i+1}$$

(14)

The automorphism C is quite useful in computations of commutators, since the action of this automorphism on strings is again a cyclic permutation:

$$C: S_i(r) \to S_{i+1}(r)$$

(15)

The obvious questions one can ask about the Lie algebra $\mathscr{A}_N$ are: What is a basis of this algebra and what are commutation relations among elements of this basis? These questions are answered in Sections 4 and 5. There we establish the isomorphism between $\mathscr{A}_N$ and the Lie algebra $\tilde{\mathscr{A}}_N$ which we define and describe in the next section.

3. THE LOOP ALGEBRA $\mathscr{L}(sl(N))$, ITS INVOLUTION, AND THE FIXED-POINT SUBALGEBRA $\tilde{\mathscr{A}}_N$

As we shall see later, the Lie Algebra $\mathscr{A}_N$ introduced in the previous section is closely related to $\mathscr{L}(sl(N))$, the $sl(N)$ loop algebra. In this section we describe a certain involution ω of $\mathscr{L}(sl(N))$ and the Lie subalgebra $\tilde{\mathscr{A}}_N$ of $\mathscr{L}sl(N))$ on which the action of ω is reduced to the identity ("fixed-point subalgebra" of ω). In subsequent sections we prove thatL the algebras $\mathscr{A}_N$ and $\tilde{\mathscr{A}}_N$ are isomorphic and we describe this isomorphism.

The $sl(N)$ loop algebra $\mathscr{L}(sl(N)) = \mathbf{C}[t, t^{-1}] \otimes sl(N)$ has the basis $\{E_{ij}^{(n)}, H_k^{(n)}\}$, $1 \leqslant i \neq j \leqslant N$; $1 \leqslant k \leqslant N-1$; $n = 0, \pm 1, \pm 2,...$. In the $N \times N$ matrix realization of $sl(N)$ the elements of this basis have the explicit form

$$E_{ij}^{(n)} = t^n E_{ij}, \qquad H_k^{(n)} = t^n H_k = t^n(E_{kk} - E_{k+1,k+1})$$

where E_{kl} is an $N \times N$ matrix all of whose entries are zero except entry (kl), which is equal to 1. The loop algebra has a linear involutive automorphism ω, $\omega^2 = id$, given by

$$\omega: \quad E_{ij}^{(n)} \to (-1)^{i+j+1+nN} E_{ji}^{(-n)} \tag{16}$$

$$\omega: \quad H_i^{(n)} \to (-1)^{1+nN} H_i^{(-n)} \tag{17}$$

This involutive automorphism ω is a product of two involutions: ω_1 and ω_2. The first of these is an involution of the algebra of Laurent polynomials:

$$\omega_1: \quad t^n \to (-1)^{nN} t^{-n}$$

The second one is an involution of $sl(N)$:

$$\omega_2: \quad E_{ij} \to (-1)^{i+j+1} E_{ji}$$

$$\omega_2: \quad H_i \to -H_i$$

It is easy to convince oneself that the subspace $\tilde{\mathscr{A}}_N \in \mathscr{L}(sl(N))$ on which the involution ω acts as identity operator is a Lie subalgebra. We can easily find the basis of this fixed-point subalgebra $\tilde{\mathscr{A}}_N$; it is formed by vectors $\{A_{ij}^{(n)}, G_i^{(n)}\}$:

$$A_{ij}^{(n)} = E_{ij}^{(n)} + (-1)^{i+j+1+nN} E_{ji}^{(-n)},$$
$$1 \leqslant i < j \leqslant N, \quad n = 0, \pm 1, \pm 2,... \tag{18}$$

$$G_i^{(n)} = H_i^{(n)} + (-1)^{nN+1} H_i^{(-n)},$$
$$1 \leqslant i \leqslant N-1, \quad n = 1, 2,... \tag{19}$$

The commutation relations of $\tilde{\mathscr{A}}_N$ in the basis $\{A_{ij}^{(n)}, G_i^{(n)}\}$ follow immediately from the commutation relations of $\mathscr{L}(sl(N))$:

$$
\begin{aligned}
[A_{ij}^{(m)}, A_{kl}^{(n)}] &= \delta_{jk} A_{il}^{(m+n)} - \delta_{il} A_{kj}^{(m+n)} \\
&\quad + \delta_{ik}(-1)^{i+j+1+mN} \theta(j < l) A_{jl}^{(n-m)} \\
&\quad + \delta_{ik}(-1)^{i+l+nN} \theta(l < j) A_{lj}^{(m-n)} \\
&\quad + \delta_{jl}(-1)^{k+l+1+nN} \theta(i < k) A_{ik}^{(m-n)} \\
&\quad + \delta_{jl}(-1)^{i+l+mN} \theta(k < i) A_{ki}^{(n-m)} \\
&\quad + \delta_{ik}\delta_{jl}(-1)^{i+j+1+nN} \sum_{s=i}^{j-1} G_s^{(m-n)}, \qquad m \geqslant n \tag{20}
\end{aligned}
$$

$$[G_i^{(m)}, A_{kl}^{(n)}] = (\delta_{ik} - \delta_{ki+1} - \delta_{li} + \delta_{li+1})$$
$$\times (A_{kl}^{(m+n)} - (-1)^{mN} A_{kl}^{(n-m)}) \qquad (21)$$

$$[G_i^{(m)}, G_j^{(n)}] = 0 \qquad (22)$$

Here δ_{ik} is the Kronecker symbol and $\theta(x)$ is the following function: $\theta(x) = 1$ (0) if x is true (false).

We will also need another basis in the algebra $\tilde{\mathscr{A}}_N$. We shall denote the elements of this new basis by symbols $\tilde{S}_i(r)$, where $r = 1, 2,...,$ and $1 \leqslant i \leqslant N$ if r is not divisible by N, and $1 \leqslant i \leqslant N - 1$ if $r = Nm$ for some positive integer m. The explicit expressions for the elements $\tilde{S}_i(r)$ are as follows:

$$\tilde{S}_i(k) = \begin{cases} E_{i,i+k} + (-1)^{k+1} E_{i+k,i} & \text{if } i+k \leqslant N \\ tE_{i,i+k-N} + (-1)^{k+1} t^{-1} E_{i+k-N,i} & i+k \geqslant N+1 \end{cases} \qquad (23)$$

for $1 \leqslant k \leqslant N - 1$;

$$\tilde{S}_i(Nm+k) = \begin{cases} [t - (-1)^N t^{-1}][t + (-1)^N t^{-1}]^{m-1} \\ \quad \times [E_{i,i+k} + (-1)^k E_{i+k,i}] & \text{if } i+k \leqslant N \\ [t - (-1)^N t^{-1}][t + (-1)^N t^{-1}]^{m-1}[tE_{i,i+k-N} \\ \quad + (-1)^k t^{-1} E_{i+k-N,i}] & \text{if } i+k \geqslant N+1 \end{cases}$$
$$(24)$$

for $1 \leqslant k \leqslant N - 1$, $m \geqslant 1$;

$$\tilde{S}_i(Nm) = [t - (-1)^N t^{-1}][t + (-1)^N t^{-1}]^{m-1} (E_{ii} - E_{i+1,i+1}) \qquad (25)$$

for $m \geqslant 1$, $1 \leqslant i \leqslant N$. Note that the elements defined by the last formula are linearly dependent: $\tilde{S}_1(Nm) + \tilde{S}_2(Nm) + \cdots + \tilde{S}_N(Nm) = 0$.

We can express the elements of the basis $\{A_{ij}^{(n)}, G_i^{(n)}\}$ in terms of the basis $\{\tilde{S}_i(r)\}$ with the aid of the recursion relations

$$A_{ij}^{(0)} = \tilde{S}_i(j-i)$$

$$A_{ij}^{(-1)} = (-1)^{1+N+i+j} \tilde{S}_j(N+j-i)$$

$$A_{ij}^{(m)} = (-1)^N A_{ij}^{(m-2)} + \Phi_i(Nm, j-i)$$
$$- (-1)^{i+j} \Phi_j(N(m-1), N+i-j)$$

$$A_{ij}^{(-m-1)} = (-1)^N A_{ij}^{(-m+1)} + (-1)^{mN} \Phi_i(N(m-1), j-i)$$
$$- (-1)^{(m+1)N+i+j} \Phi_j(Nm, N+i-j)$$

$$G_i^{(m)} = \Phi_i(Nm, 0)$$

for $m \geqslant 1$. The vectors $\Phi_i(Nm, s)$ are given in terms of $\tilde{S}_i(r)$ by the formula

$$\Phi_i(Nm, s) = \sum_{r=1}^{m} c(m, r) \, \tilde{S}_i(Nr + s), \qquad 0 \leqslant s \leqslant N - 1$$

where the coefficients $c(m, r)$ are defined by the recursion relation:

$$c(m + 1, r) = (1 - \delta_{1r}) \, c(m, r - 1) - (1 - \delta_{mr})(1 - \delta_{m+1r})(-1)^N \, c(m - 1, r)$$

$$c(1, 1) = 1$$

The basis of vectors $\{\tilde{S}_i(r)\}$ which we described in this section will be used in the proof of an isomorphism of the algebras $\mathscr{A}_N$ and $\tilde{\mathscr{A}}_N$.

4. STRUCTURE OF THE ALGEBRA $\mathscr{A}_N$

In this section we study the structure of the algebra $\mathscr{A}_N$ in some detail. The results obtained here will be extensively used in Section 5 to establish isomorphism of the algebras $\mathscr{A}_N$ and $\tilde{\mathscr{A}}_N$. The main result that we establish is formulated as the following proposition:

Proposition 1. The Lie algebra $\mathscr{A}_N$ is spanned by the set of strings $\{S_i(r)\}$, $1 \leqslant i \leqslant N$, $r = 1, 2, \ldots$, as a linear space.

Notice that it is not true that all strings are linearly independent.

In order to prove this proposition we shall first compute commutators of the generators of $\mathscr{A}_N$ with all strings, that is, the commutators of the form

$$[\mathbf{e}_i, S_j(r)] \overset{\text{def}}{=} [i, S_j(r)]$$

where $1 \leqslant i$, $j \leqslant N$ and $r = 1, 2, \ldots$. Due to the existence of the cyclic automorphism C it is sufficient to compute $[1, S_j(r)]$ for all j and r; the rest of the commutators $[i, S_j(r)]$ are then immediately obtained by application of C. The result which we get computing $[1, S_j(r)]$ is summarized in the following lemma. The distinctive feature of the strings which emerges from the result of the lemma is that a commutator of a generator with a string is again a string.

Lemma 1. The following relations hold in $\mathscr{A}_N$ for $m \geqslant 0$: $[1, S_k(Nm + r)] =$

1. If $1 \leqslant k \leqslant N$, $1 \leqslant r \leqslant N-1$, and $k+r \leqslant N$,

     (a_m) $-2S_1(Nm)$ when $k=1, r=1$

     (b_m) $S_2(Nm+r-1)$ when $k=1, r \geqslant 2$

     (c_m) $S_1(Nm+r+1)$ when $k=2$

     (d_m) 0 when $k \geqslant 3$

2. If $1 \leqslant k \leqslant N$, $1 \leqslant r \leqslant N-1$, and $k+r \geqslant N+1$,

     (e_m) $-S_k(Nm+r+1)$ when $k+r=N+1, k \neq 2$

     (f_m) $S_1(Nm+r+1)$ when $k+r=N+1, k=2$

     (g_m) $-S_k(Nm+r-1)$ when $k+r=N+2$

     (h_m) 0 when $k+r \geqslant N+3$

3. If $1 \leqslant k \leqslant N$ and $r=N$,

     (i_m) $-2S_1(Nm+N+1)$ when $k=1$

     (j_m) $S_1(Nm+N+1)$ when $k=2$

     (k_m) $S_1(Nm+N+1)$ when $k=N$

     (l_m) 0 when $3 \leqslant k \leqslant N-1$

Proof. We shall prove the lemma using induction in m. First we establish the base of the induction by proving relations a_0 through l_0, and then show that relations a_m through l_m entail a_{m+1} through l_{m+1}. At each elementary step we employ either the defining relations of $\mathscr{A}_N$ or the Jacobi identity or skew symmetry of the commutator. The proof given below is literally valid for $N \geqslant 4$. The idea of the proof for $N=3$ is completely the same, while precise wording (e.g., ranges of variation of various subindices) of some of the statements encountered in the course of the proof given below needs to be slightly changed. Since these changes are obvious and do not require any new ideas as compared to the case $N \geqslant 4$, we omit the proof for the case $N=3$.

1. Proof of the induction base. We compute the commutator $[1, S_k(r)]$ when $1 \leqslant k \leqslant N$ and $1 \leqslant r \leqslant N$.

Case a_0. $k=1$, $r=1$: $[1, S_1(1)] =^{\text{def}} [1, 1] = 0 =^{\text{def}} -2S_1(0)$

Case d_0. $k \geqslant 3$, $k+r \leqslant N$, $r \leqslant N-1$:

$$x \overset{\text{def}}{=} [1, S_k(r)] \overset{\text{def}}{=} [1, k, k+1,..., k+r-1]$$

Since $k \geqslant 3$ and $k+r \leqslant N-1$, 1 commutes with all k, $k+1,\ldots$, $k+r-1$. Hence $x = 0$.

Case b_0. $k=1$, $r \geqslant 2$, $k+r \leqslant N$:

$$x \stackrel{\text{def}}{=} [1, S_k(r)] \stackrel{\text{def}}{=} [1, 1, 2,\ldots, r]$$

Since $r \leqslant N-1$

$$x = [1, [1, 2], 3,\ldots, r] = [2, 3,\ldots, r] \stackrel{\text{def}}{=} S_2(r-1)$$

Case h_0. $k+r \geqslant N+3$, $r \leqslant N-1$:

$$x \stackrel{\text{def}}{=} [1, S_k(r)] \stackrel{\text{def}}{=} [1, k, k+1,\ldots, N, 1, 2,\ldots, k+r-1-N]$$

Since $2 \leqslant k+r-1-N \leqslant N-2$,

$$x = [k, k+1,\ldots, N-1, [1, N], 1, 2,\ldots, k+r-1-N]$$
$$+ [k, k+1,\ldots, N, 1, 1, 2,\ldots, k+r-1-N]$$

Denote the first (second) summand in the right-hand side of the above formula by a (b). Then we find that

$$a = -[k, k+1,\ldots, N, 2,\ldots, k+r-1-N]$$
$$+ [k, k+1,\ldots, N-1, 1, [1, N], 2,\ldots, k+r-1-N]$$
$$= -[k, k+1,\ldots, N, 2,\ldots, k+r-1-N]$$
$$+ [k, k+1,\ldots, N-1, 1, 1, N, 2,\ldots, k+r-1-N]$$
$$- [k, k+1,\ldots, N-1, 1, N, 1, 2,\ldots, k+r-1-N]$$

The first two summands above and the commutator b vanish since N commutes with all elements standing on its right in these expressions. Hence $a = -x$, because the leftmost element 1 in the third summand above commutes with all elements standing on its left. Therefore $x = a + b = -x$ and $x = 0$.

Case e_0. $k+r = N+1$, $k \geqslant 3$, $r \leqslant N-1$:

$$[1, S_k(r)] \stackrel{\text{def}}{=} [1, k, k+1,\ldots, N-1, N]$$
$$= -[k, k+1,\ldots, N-1, N, 1] \stackrel{\text{def}}{=} -S_k(r+1)$$

Case g_0. $k + r = N + 2$, $r \leqslant N - 1$:

$$[1, S_k(r)] \overset{\text{def}}{=} [1, k, k+1,..., N-1, N, 1]$$
$$= -[k, k+1,..., N-1, 1, 1, N]$$
$$= -[k, k+1,..., N-1, N] \overset{\text{def}}{=} -S_k(r-1)$$

Case i_0. $k = 1$, $r = N$:

$$x \overset{\text{def}}{=} [1, S_1(N)] \overset{\text{def}}{=} [1, 1, 2,..., N]$$
$$= [1, [1, 2], 3,..., N] - [1, 2,..., N, 1]$$

Denote the first (second) summand in the right-hand side of the above formula by a ($-b$). Then

$$a = [2, 3,..., N] + [[1, 2], 1, 3,..., N]$$
$$= [2, 3,..., N] - [1, 2, 3,..., N, 1] + [2, 1, 3,..., N, 1]$$
$$= -[1, 2, 3,..., N, 1] = -b$$

Hence

$$x = a - b = -2[1, 2, 3,..., N, 1] \overset{\text{def}}{=} -2S_1(N+1)$$

Case k_0. $k = N$, $r = N$:

$$x \overset{\text{def}}{=} [1, S_N(N)] \overset{\text{def}}{=} [1, N, 1, S_2(N-2)] = a + b$$

where $a = [[1, N], 1, S_2(N-2)]$ and $b = [N, 1, S_1(N-1)]$. Using the defining relations of $\mathscr{A}_N$ and the Jacobi identity, we find that

$$a = -[N, S_2(N-2)] + [1, 1, N, S_2(N-2)] - [1, N, 1, S_2(N-2)]$$

Using the already proven relation e_0 and the automorphism C, one finds that $[N, S_2(N-2)] = -S_2(N-1)$. Hence $a = S_2(N-1) - [1, S_1(N)] - [1, S_N(N)]$. The relation i_0 then leads to $a = S_2(N-1) + 2S_1(N+1) - x$. The already proven relation b_0 gives $b = [N, S_2(N-2)] = -S_2(N-1)$. Hence we obtain $x = a + b = 2S_1(N+1) - x$, and $x = S_1(N+1)$.

Case l_0. $3 \leqslant k \leqslant N-1$, $r = N$:

$$[1, S_k(N)] = [1, k, S_{k+1}(N-1)] = [k, 1, S_{k+1}(N-1)] = 0$$

since the internal commutator in the last formula vanishes due to the already proven relation h_0.

The rest of the cases, i.e., c_0, f_0, and j_0, are immediate by the definition of the elements $S_i(r)$.

The induction base is proven.

2. Now we prove the induction step: the relations $a_{m+1},...,l_{m+1}$ follow from the relations $a_m,...,l_m$.

Case a_{m+1}. $k = 1$, $r = 1$:

$$x \stackrel{\text{def}}{=} [1, S_1(N(m+1)+1] \stackrel{\text{def}}{=} [1, 1, 2, S_3(Nm+N-1)]$$

$$= [1, [1, 2], S_3(Nm+N-1)] + [1, 2, 1, S_3(Nm+N-1)]$$

Denote the first (second) summand in the right-hand side of the last formula by a (b). Then using the defining relations and the Jacobi identity, we obtain

$$a = [2, S_3(Nm+N-1)] + [[1, 2], 1, S_3(Nm+N-1)]$$

Denoting the second summand in the last formula by a_2 and using the identity g_m to compute the commutator $[1, S_3(Nm+N-1)]$, we arrive at the equation

$$a_2 = -[1, 2, S_3(Nm+N-2)] + [2, 1, S_3(Nm_N-2)]$$

Now we apply the identity e_m together with the definition of a string and find that $a_2 = -S_1(Nm+N) - [2, S_3(Nm+N-1)]$.

Using g_m, one gets $b = -S_1(Nm+N)$. Putting expressions for a and b together, we find

$$x = a + b = -2S_1(N(m+1))$$

Case d_{m+1}. $k \geqslant 3$, $k+r \leqslant N$, $r \leqslant N-1$:

$$x \stackrel{\text{def}}{=} [1, S_k(N(m+1)+r)] \stackrel{\text{def}}{=} [1, k, k+1,..., N, 1, 2,..., k-1, S_k(Nm+r)]$$

The relation d_m gives

$$x = [k, k+1,..., N-1, [1, N], 1, 2,..., k-1, S_k(Nm+r)]$$

$$+ [k, k+1,..., N, 1, [1, 2], 3,..., k-1, S_k(Nm+r)]$$

Denote the first (second) summand in x by a (b). Then the defining relations, the Jacobi identity, and d_m give

$$b = [k, k+1,..., N, 2, 3,..., k-1, S_k(Nm+r)]$$

$$a = -b + [k, k+1,..., N-1, 1, [1, N], 2, 3,..., k-1, S_k(Nm+r)]$$

For x we obtain

$$x = a+b = [k, k+1,..., N-1, 1, 1, N, 2, 3,..., k-1, S_k(Nm+r)]$$
$$- [k, k+1,..., N-1, 1, N, 1, 2,..., k-1, S_k(Nm+r)]$$

Since in the second summand above the leftmost 1 commutes with all elements standing on its left, we can carry this generator 1 to the very left. In the first summand above, N commutes with all elements standing on its right up to $S_k(Nm+r)$. Hence we obtain

$$2x = [k, k+1,..., N-1, 1, 1, 2, 3,..., k-1, N, S_k(Nm+r)]$$

Let us now compute the commutator $[N, S_k(Nm+r)]$ standing to the very right in the above expression. The identities d_m, e_m together with the application of the automorphism C give $[N, S_k(Nm+r)] = -\delta_{r+k,N} S_k(Nm+r+1)$. Therefore

$$2x = -\delta_{r+k,N}[k, k+1,..., N-1, 1, 1, 2,..., k-1, S_k(Nm+r+1)]$$
$$= -\delta_{r+k,N}[k, k+1,..., N-1, 1, S_1(Nm+r+k)]$$
$$= -\delta_{r+k,N}[k, k+1,..., N-1, 1, N-1, S_1(Nm+N)]$$

The relation 1_m applied (together with the automorphism C) to the commutator $[N-1, S_1(Nm+N)]$ gives $x = 0$.

Case b_{m+1}. $k=1$, $r \geqslant 2$, $k+r \leqslant N$:

$$[1, S_1(N(m+1)+r)] = [1, 1, 2, S_3(N(m+1)+r-2))]$$
$$= [1, [1, 2], S_3(N(m+1)+r-2)]$$
$$+ [1, 2, 1, S_3(N(m+1)+r-2)]$$

The commutator $[1, S_3(N(m+1)+r-2)]$ is equal to zero either because of the already proven relation d_{m+1} or, when $r=2$, because of the relation 1_m. Consequently

$$[1, S_1(N(m+1)+r)] = [2, S_3(N(m+1)+r-2)] \stackrel{\text{def}}{=} S_2(N(m+1)+r+1)$$

Case h_{m+1}. $k+r \geqslant N+3$, $r \leqslant N-1$:

$$x \stackrel{\text{def}}{=} [1, S_k(N(m+1)+r)]$$
$$= [1, k, k+1,..., N, 1, 2,..., k-1, S_k(Nm+r)]$$

By virtue of the Jacobi identity,

$$x = [k, k+1,..., N-1, [1, N], 1, 2,..., k-1, S_k(Nm+r)]$$
$$+ [k, k+1,..., N-1, N, 1, 1, 2,..., k-1, S_k(Nm+r)]$$

Let us denote the first (second) summand in the right-hand side of the above formula by a (b). Due to the Jacobi identity and the defining relations,

$$a = -[k, k+1,..., N-1, N, 2, 3,..., k-1, S_k(Nm+r)]$$
$$+ [k, k+1,..., N-1, 1, [1, N], 2, 3,..., k-1, S_k(Nm+r)]$$

Denoting the first (second) summand in the right-hand side of the above formula by a_1 (a_2) and using the definition of a string, we find $a_1 = [k, k+1,..., N-1, N, S_2(Nm+r+k-2)]$. Then applying the already proven relation d_{m+1} and the automorphism C to compute the commutator $[N, S_2(Nm+r+k-2)]$, we obtain $a_1 = 0$. For a_2 we have

$$a_2 = [k, k+1,..., N-1, 1, 1, N, S_2(Nm+r+k-2)]$$
$$- [k, k+1,..., N-1, 1, N, 1, 2,..., k-1, S_k(Nm+r)]$$

The first term in this expression for a_2 is equal to zero because $[N, S_2(Nm+r+k-2)] = 0$, while the second term is equal to $-x$.

Applying the already proven relation b_{m+1} and the same reasoning as in the computation of a_1, we find that $b = 0$. Therefore $x = a_1 + a_2 + b = -x$; $x = 0$.

Case e_{m+1}. $k+r = N+1$, $k \geqslant 3$, $r \leqslant N-1$.
Applying the Jacobi identity, we obtain

$$x \stackrel{\text{def}}{=} [1, S_k(N(m+1)+r)]$$
$$\stackrel{\text{def}}{=} [1, k, k+1,..., N, 1, 2,..., k-1, S_k(Nm+r)]$$
$$= [k, k+1,..., N-1, [1, N], 1, 2,..., S_k(Nm+r)]$$
$$+ [k, k+1,..., N, 1, [1, 2], 3,..., k-1, S_k(Nm+r)]$$
$$+ [k, k+1,..., N-1, N, 1, 2,..., k-1, 1, S_k(Nm+r)]$$

Let us denote the three summands standing in the right-hand side of the last equality in the above formula by a, b, c. Using the defining relations, the Jacobi identity, and the definition of a string, we come to the equality

$$a = -[k, k+1,..., N-1, N, S_2(Nm+r+k-2)]$$
$$+ [k, k+1,..., N-1, 1, 1, N, 2, 3,..., S_k(Nm+r)] - x$$

Denoting the first (second) summand in the above expression for a by a_1 (a_2) and using the relation g_m, we find that $a_1 = [k, k+1,..., N-1, S_2(Nm+N-2)]$, whereas applying the definition of a string and the relation b_m, we find that $a_2 = -a_1$. Therefore $a = -x$.

For b we obtain

$$b \overset{\text{def}}{=} [k, k+1,..., N, 1, [1, 2],..., k-1, S_k(Nm+r)]$$
$$= [k, k+1,..., N, 1, 1, 2,..., k-1, S_k(Nm+r)]$$
$$- [k, k+1,..., N, 1, 2, 1, 3,..., k-1, S_k(Nm+r)]$$

Denote the first (second) summand in the right-hand side of the above equality by b_1 $(-b_2)$. Then the definition of a string and i_m enable us to write

$$b_1 = [k, k+1,..., N, 1, S_1(Nm+r+k-1)]$$
$$= -2[k, k+1,..., N, S_1(Nm+N+1)]$$
$$= -2S_k(N(m+1)+r+1)$$

whereas e_m applied to the commutator $[1, S_3(Nm+r+k-3)]$ entering b_2 gives $b_2 = -S_k(N(m+1)+r+1)$.

The relation e_m applied to the commutator $[1, S_k(Nm+r)]$ entering c leads to $c = b_2$. Finally,

$$x = a + b + c = -x - 2S_k(N(m+1)+r-1)$$
$$x = -S_k(N(m+1)+r+1)$$

Case g_{m+1}. $k+r = N+2$, $r \leqslant N-1$.
Applying the Jacobi identity, we obtain

$$x \overset{\text{def}}{=} [1, S_k(N(m+1)+r)]$$
$$\overset{\text{def}}{=} [1, k, k+1,..., N, 1, 2,..., k-1, S_k(Nm+r)]$$
$$= [k, k+1,..., N-1, [1, N], 1, 2,..., S_k(Nm+r)]$$
$$+ [k, k+1,..., N, 1, [1, 2], 3,..., k-1, S_k(Nm+r)]$$
$$+ [k, k+1,..., N-1, N, 1, 2,..., k-1, 1, S_k(Nm+r)]$$

Let us again denote the three summands standing in the right-hand side of the last equality in the above formula by a, b, c. Using the defining relations, the Jacobi identity, and the definition of a string, we come to the equality

$$a = -[k, k+1,..., N-1, N, S_2(Nm+r+k-2)]$$
$$+ [k, k+1,..., N-1, 1, 1, N, S_2(Nm+k+r-2)] - x$$

Using the relation l_m and the automorphism C to compute the commutator $[N, S_2(Nm+r+k-2)]$, we arrive at $a = -x$.

For b we obtain

$$b \stackrel{\text{def}}{=} [k, k+1,..., N, 1, [1, 2], 3,..., k-1, S_k(Nm+r)]$$
$$= [k, k+1,..., N, 1, 1, 2,..., k-1, S_k(Nm+r)]$$
$$- [k, k+1,..., N, 1, 2, 1, 3,..., k-1, S_k(Nm+r)]$$
$$= [k, k+1,..., N, 1, S_1(Nm+N+1)]$$
$$- [k, k+1,..., N, 1, 2, 1, S_3(Nm+N-1)]$$

Now we use the already proven relation a_{m+1} to compute the commutator $[1, S_1(Nm+N+1)]$ inside the first bracket above and the relation g_m to compute the commutator $[1, S_3(Nm+N-1)]$ inside the second one. This gives $b = -S_k(N(m+1)+r-1)$.

Applying g_m to the commutator $[1, S_k(Nm+r)]$ entering c, we get $c = b$. Finally,

$$x = a + b + c = -x - 2S_k(N(m+1)+r-1)$$
$$x = -S_k(N(m+1)+r-1)$$

Case i_{m+1}. $k = 1$, $r = N$:

$$x \stackrel{\text{def}}{=} [1, S_1(N(m+1)+N)]$$
$$\stackrel{\text{def}}{=} [1, 1, 2, S_3(N(m+1)+N-2)]$$
$$= [1, [1, 2], S_3(N(m+1)+N-2)] + [1, 2, 1, S_3(N(m+1)+N-2)]$$

Denoting the first (second) summand in the right-hand side of the last formula by a (b) and using the defining relations, we obtain

$$a = [2, S_3(N(m+1)+N-2)] + [[1, 2], 1, S_3(N(m+1)+N-2)]$$

Applying the already proven relation e_{m+1} to compute the commutator $[1, S_3(N(m+1)+N-2)]$, one gets

$$a = [2, S_3(N(m+1)+N-2)] - S_1(N(m+1)+N+1)$$
$$+ [2, 1, S_3(N(m+1)+N-1)]$$

Transforming the last summand with the aid of the already proven relation g_{m+1}, we arrive at $a = -S_1(N(m+1)+N+1)$.

Application of e_{m+1} to b gives $b = a$. Hence $x = a + b = -2S_1(N(m+1)+N+1)$.

Case 1_{m+1}. $3 \leqslant k \leqslant N-1$, $r = N$:

$$[1, S_k(N(m+1)+N)] = [k, 1, S_{k+1}(N(m+1)+N-1)] = 0$$

In virtue of the already proven relation h_{m+1}.

Case k_{m+1}. $k = N$, $r = N$:

$$x \stackrel{\text{def}}{=} [1, S_N(N(m+1)+N)] = [1, N, 1, S_2(N(m+1)+N-2$$
$$= [[1, N], 1, S_2(N(m+1)+N-2)]$$
$$+ [N, 1, 1, S_2(N(m+1)+N-2)]$$

Denoting the first (second) summand in the right-hand side of the last equality by a (b) and using the defining relations, we obtain

$$a = -[N, S_2(N(m+1)+N-2] + [1, 1, N, S_2(N(m+1)+N-2)] - x$$

Transforming the second summand in the above expression for a with the aid of the already proven relations e_{m+1} and i_{m+1}, we arrive at

$$a = -[N, S_2(N(m+1)+N-2)] + 2S_2(N(m+1)+N+1) - x$$

For b we have $b = [N, 1, S_1(N(m+1)+N+1)]$. Taking into account the already proven relation b_{m+1}, one then gets $b = [N, S_2(N(m+1)+N-2]$. Finally,

$$x = a + b = -x + 2S_1(N(m+1)+N+1)$$
$$x = S_1(N(m+1)+N+1)$$

The remaining cases $c_{m+1}, f_{m+1}, j_{m+1}$, are direct consequences of the definition of the strings.

Thus the induction step is proven. ▌

The lemma has an obvious corollary:

Corollary. The elements $x(m) \overset{\text{def}}{=} \sum_{i=1}^{N} S_i(Nm)$ belong to the center of $\mathscr{A}_N$.

Proof. It follows at once from the statements i_m, j_m, k_m and l_m of the lemma by appication of the automorphism C that $[i, x(m)] = 0$, $1 \leqslant i \leqslant N$, $m \geqslant 1$. ∎

Now we can proceed further and turn to the proof of Proposition 1. First of all we notice that any multiple commutator of $\mathbf{e}_i$'s can be converted with the aid of the Jacobi identity into a linear combination of commutators nested to the right, i.e., commutators of the form $[i_1, i_2, i_3,..., i_{m-1}, i_m]$ for some set of $1 \leqslant i_k \leqslant N$. Let us compute the nested commutators in the last expression starting from the innermost one, $[i_{m-1}, i_m]$, and going step by step outward. At each step of this procedure we need to compute a commutator of a generator with a string which is, according to Lemma 1, again a string. Therefore any commutator of the form $[i_1, i_2, i_3,..., i_{m-1}, i_m]$ is a string (it may be of zero length, then it is equal to 0). Thus any multiple commutator of the generators is a linear combination of strings (with integer coefficients). This finishes the proof of Proposition 1.

In principle now we could find commutation relations among all strings using the Jacobi identity and the result of Lemma 1. Such a computation, though, is rather cumbersome and we did it only for $N = 3$. For arbitrary $N \geqslant 3$ in Section 5 we follow another route which eventually gives a basis of the algebra $\mathscr{A}_N$ in terms of strings and commutation relations between the elements of this basis.

5. ISOMORPHISM OF THE ALGEBRAS $\mathscr{A}_N$ AND $\tilde{\mathscr{A}}_N$

In this section we show that the algebra $\mathscr{A}_N$ defined in Section 2 and the algebra $\tilde{\mathscr{A}}_N$ defined in Section 3 are isomorphic. Lemma 1 and Proposition 1 of the previous section are the main ingredients used to show the isomorphism.

We define a linear map from $\mathscr{A}_N$ to $\tilde{\mathscr{A}}_N$ which we call π. First, we define this map on the generators of $\mathscr{A}_N$ as follows:

$$\pi(\mathbf{e}_i) \overset{\text{def}}{=} \tilde{S}_i(1), \qquad 1 \leqslant i \leqslant N \tag{26}$$

where the vectors $\tilde{S}_i(1)$ were defined in (23) in Section 2.

Next, we define the map π on the whole algebra $\mathscr{A}_N$ by the prescription

$$\pi([a, b]) \overset{\text{def}}{=} [\pi(a), \pi(b)] \tag{27}$$

It is easy to check that the vectors $\pi(\mathbf{e}_i)$ satisfy the relations (11)–(12). Therefore the map $\pi: \mathscr{A}_N \to \tilde{\mathscr{A}}_N$ is a homomorphism of Lie algebras. Now we wish to find out what is the image of the algebra $\mathscr{A}_N$ under the action of π. Since $\mathscr{A}_N$ is spanned by strings, it is sufficient to find images of all strings, i.e., vectors $\pi(S_i(r))$, $1 \leqslant i \leqslant N$. Using the definition of a string and the prescription (27), we arrive at the recursion relation

$$\pi(S_i(r)) = [\pi(\mathbf{e}_i), \pi(S_{i+1}(r-1))] \tag{28}$$

This recusion relation is supplemented by the initial conditions (26); therefore we can solve it, the result being

$$\pi(S_i(r)) = \tilde{S}_i(r), \qquad 1 \leqslant i \leqslant N, \quad r = 1, 2, \ldots \tag{29}$$

where $\tilde{S}_i(r)$ are defined in (23)–(25). Since $\tilde{\mathscr{A}}_N$ is a linear span of the vectors $\tilde{S}_i(r)$, $1 \leqslant i \leqslant N$, $r = 1, 2, \ldots$, we come to the conclusion that the image of $\mathscr{A}_N$ is the whole algebra $\tilde{\mathscr{A}}_N$: $\operatorname{Im} \pi|_{\mathscr{A}_N} = \tilde{\mathscr{A}}_N$. Now we notice that all the vectors $\tilde{S}_i(r)$ $(1 \leqslant i \leqslant N, r = 1, 2, \ldots)$ are linearly independent in $\tilde{\mathscr{A}}_N$ except that $\sum_{i=1}^{N} \tilde{S}_i(Nm) = 0$ for all $m \geqslant 1$. Therefore we come to the conclusion that the kernel of the homomorphism π is given by

$$\operatorname{Ker} \pi = \text{linear span of } \{x(m)\}_{m \geqslant 1} \qquad (\overset{\text{def}}{=} \mathbf{C}\{x(m)\}_{m \geqslant 1}) \tag{30}$$

where the elements $x(m) = ^{\text{def}} \sum_{i=1}^{N} S_i(Nm)$ were defined in the corollary to Lemma 1. Recall that according to this corollary, $\mathbf{C}\{x(m)\}_{m \geqslant 1}$ belongs to the center of the Lie algebra $\mathscr{A}_N$. Hence we conclude that $\mathscr{A}_N$ is isomorphic to a central extension of $\tilde{\mathscr{A}}_N$ by $\mathbf{C}\{x(m)\}_{m \geqslant 1}$. In order to prove that the map π is an isomorphism, we have to show that $\operatorname{Ker} \pi = 0$.

Now we have the following proposition:

Proposition 2. For all $m \geqslant 1$ the central elements $x(m) = ^{\text{def}} \sum_{i=1}^{N} S_i(Nm)$ vanish.

Proof. To prove this proposition we first compute the commutator $C(l, m) = ^{\text{def}} [S_1(Nl), S_1(Nm)]$ for $l, m \geqslant 1$.

In virtue of Lemma 1, one has for $m \geqslant 1$

$$[k, S_1(Nm+k-1)] = \begin{cases} -2S_1(Nm+1) & \text{if} \quad k = 1 \\ -S_1(Nm+k) & \text{if} \quad 2 \leqslant k \leqslant N-1 \\ S_N(Nm+N) & \text{if} \quad k = N \end{cases} \tag{31}$$

Applying the above equality and the Jacobi identity, we then get for l, $m \geqslant 1$, $1 \leqslant k \leqslant N-1$,

$$[S_k(Nl-k+1), S_1(Nm+k-1)]$$
$$= [k, S_{k+1}(Nl-k), S_1(Nm+k-1)]$$
$$+ (1+\delta_{k,1})[S_{k+1}(Nl-k), S_1(Nm+k)] \tag{32}$$

Using the last formula repeatedly, we arrive at the equality

$$C(l, m) = [1, S_2(Nl-1), S_1(Nm)]$$
$$+ 2\sum_{k=2}^{N-1} [k, S_{k+1}(Nl-k), S_1(Nm+k-1)]$$
$$+ 2[S_N(Nl-N+1), S_1(Nm+N-1)] \tag{33}$$

Equation (31) leads to the following equality for $m \geqslant 1$:

$$[S_N(Nl-N+1), S_1(Nm+N-1)]$$
$$= \begin{cases} S_N(N(m+1)) & \text{if } l=1 \\ [N, S_1(Nl-N), S_1(Nm+N-1)] & \\ -[S_1(Nl-N), S_N(Nm+N)] & \text{if } l\geqslant 2 \end{cases} \tag{34}$$

Combining the relations (33) and (34), we come to

$$C(1, m) = [1, S_2(N-1), S_1(Nm)]$$
$$+ 2\sum_{k=2}^{N-1} [k, S_{k+1}(N-k), S_1(Nm+k-1)]$$
$$+ 2S_N(N(m+1)), \qquad m\geqslant 1 \tag{35}$$
$$C(l, m) = [1, S_2(N-1), S_1(Nm)]$$
$$+ 2\sum_{k=2}^{N} [k, S_{k+1}(Nl-k), S_1(Nm+k-1)]$$
$$- 2[S_1(Nl-N), S_N(Nm+N)], \qquad m\geqslant 1, l\geqslant 2 \tag{36}$$

Moreover, Lemma 1 and the Jacobi identity also give the following equation for $l, m \geqslant 1$:

$$[S_1(Nl), S_1(Nm)] + 2[S_1(Nl), S_N(Nm)]$$
$$= [1, S_2(Nl-1), S_1(Nm)] + 2[1, S_2(Nl-1), S_N(Nm)] \tag{37}$$

The next step is to compute the triple commutators of the form $[i, S_j(p), S_k(q)]$ appearing in Eqs. (35)–(37). We shall do it as follows. First we can compute the internal commutators $[S_j(p), S_k(q)]$ up to a central

element using the values of commutators $[\tilde{S}_j(p), \tilde{S}_k(q)]$ in the algebra $\tilde{\mathscr{A}}_N$. Indeed, suppose we know that

$$[\tilde{S}_j(p), \tilde{S}_k(q)] = \sum_{l,r} F^{l,r}_{j;\,p;k,q} \tilde{S}_l(r)$$

where $F^{l,r}_{j;\,p;k,q}$ are known structure constants. Then since $\mathscr{A}_N$ is a central extension of $\tilde{\mathscr{A}}_N$ (by the linear span of the set $\{x(m)\}_{m \geqslant 1}$), we have

$$[S_j(p), S_k(q)] = \sum_{l,r} F^{l,r}_{j;\,p;k,q} S_l(r) + X$$

where X is some element in the center of $\mathscr{A}_N$.

Next, we compute $[i, S_j(p), S_k(q)]$ using the result of Lemma 1.

It is straightforward to find the commutators $[\tilde{S}_j(p), \tilde{S}_k(q)]$; their computation gives the following formulas for the relevant triple commutators for $m \geqslant 1$:

$$[1, S_2(N-1), S_1(Nm)] = 2S_1(Nm+N)$$

$$[k, S_{k+1}(N-k), S_1(Nm+k-1)] = S_k(Nm)$$

$$[1, S_2(N-1), S_N(Nm)] = -S_1(Nm+N)$$

$$[1, S_2(Nl-1), S_1(Nm)] = 2S_1(N(m+l)) - 8(-1)^N$$
$$\times S_1(N(m+l-2)), \qquad l \geqslant 2 \tag{38}$$

$$[k, S_{k+1}(Nl-k), S_1(Nm+k-1)] = S_k(N(l+m)) - 4(-1)^N$$
$$\times S_k(N(m+l-2)), \qquad l, k \geqslant 2$$

$$[1, S_2(Nl-1), S_1(Nm)] = -2[1, S_2(Nl-1), S_N(Nm)]$$

Substituting these expressions into (35)–(37), we obtain for $m \geqslant 1$

$$C(1, m) = 2 \sum_{k=1}^{N} S_k(N(m+1)) \stackrel{\text{def}}{=} 2x(m+1)$$
$$C(l, m) = C(l-1, m+1) + 2x(l+m) - 8(-1)^N x(l+m-2), \qquad l \geqslant 2 \tag{39}$$

Now taking into account that $C(l, m) + C(m, l) = 0$, we arrive at the following recursion relation for $x(m)$:

$$mx(m) = 4(m-2)(-1)^N x(m-2), \qquad m \geqslant 3 \tag{40}$$

This recursion relation is supplemented by two initial conditions: $x(2) = 0$, which follows from 39), and $x(1) = 0$, which will be shown shortly. Solving the recursion relation (40) with these initial conditions, we get the desired result: $x(m) = 0$.

Now we show that $x(1) = 0$:

$$x(1) \overset{\text{def}}{=} [1, 2,..., N] + [2, 3,..., 1] + \cdots + [N, 1,..., N-1]$$

If $N = 3$, Eq. $x(1) = 0$ is the Jacobi identity. If $N \geqslant 4$, using the defining relations of $\mathscr{A}_N$, we find

$$[N, 1,..., N-1] = [[N, 1], 2, 3,..., N-1] - [1, 2,..., N]$$

$$[[N, 1, 2,..., k], k+1, k+2,..., N-1]$$

$$= [[N, 1, 2,..., k+1], k+2,..., N-1] - [k+1,..., k],$$

$$k \leqslant N-3$$

Applying the last equation repeatedly, we arrive at

$$[N, 1, 2,..., N-1]$$

$$= -1[1, 2,..., N] - [2, 3,..., N, 1] - \cdots - [N-1, N, 1, 2,..., N-2]$$

The proposition is proven. ∎

In virtue of Proposition 2, Ker $\pi = 0$, hence the map π defined in (29) is an isomorphism of Lie algebras.

6. THE HAMILTONIAN AND THE INTEGRALS OF MOTION

Now we are ready to discuss how the $sl(N)$ Onsager's algebra leads to the existence of an infinite number of integrals of motion for a Hamiltonian which is linear in the generators $\mathbf{e}_i$ of this algebra.

Suppose that we have a representation of the algebra $\mathscr{A}_N$, i.e., a set of N linear operators $\mathbf{e}_i$, $1 \leqslant i \leqslant N$, satisfying the generalized Dolan–Grady conditions (11), (12). Consider the operator H, the Hamiltonian

$$H = k_1 \mathbf{e}_1 + k_2 \mathbf{e}_2 + \cdots + k_N \mathbf{e}_N \tag{41}$$

where k_i are some constants.

If we consider H as a vector in $\mathscr{A}_N$, then the image $\pi(H)$ of H under the action of the isomorphism π, (29), is a vector in $\tilde{\mathscr{A}}_N$:

$$\pi(H) = \sum_{i=1}^{N} k_i \tilde{S}_i(1) \tag{42}$$

$$= \sum_{i=1}^{N-1} k_i (E_{i,i+1} + E_{i+1,i}) + k_N (tE_{N1} + t^{-1}E_{1N}) \tag{43}$$

Now let us consider the following set of vectors in $\tilde{\mathscr{A}}_N$:

$$\tilde{I}_1 = \sum_{i=1}^{N} k_i(\tilde{S}_i(N+1) + 2\tilde{S}_{i+1}(N-1)) \tag{44}$$

$$= (t + (-1)^N t^{-1})\,\pi(H) \tag{45}$$

$$\tilde{I}_m = \sum_{i=1}^{N} k_i(\tilde{S}_i(Nm+1) + 2\tilde{S}_{i+1}(Nm-1)) \tag{46}$$

$$= (t - (-1)^N t^{-1})^2\,(t + (-1)^N t^{-1})^{m-2}\,\pi(H), \qquad m = 2, 3,\ldots \tag{47}$$

These vectors obviously commute between themselves and with the operator $\pi(H)$. Therefore taking the inverse of π we arrive to the conclusion that the members of the following set of elements $\{I_m\}_{m \geqslant 0}$ of the algebra $\mathscr{A}_N$ commute between themselves:

$$I_m = \sum_{i=1}^{N} k_i(S_i(Nm+1) + 2S_{i+1}(Nm-1)), \qquad m \geqslant 1 \tag{48}$$

$$I_0 = H \tag{49}$$

$$[I_m, I_n] = 0, \qquad m \geqslant 0 \tag{50}$$

Thus if the conditions (11), (12) are satisfied, H is a member of the infinite family of integrals of motion in involution. Each of these integrals is a vector in $\mathscr{A}_N$ and is expressed in terms of the operators e_i according to the definition of strings $S_i(r)$ given in (13). Since the strings are linearl independent, the vectors I_k are linearly independent in $\mathscr{A}_N$ as well.

7. CONCLUSION

As we have seen, the original Dolan–Grady relations that define Onsager's algebra admit a generalization. This generalization stands in the same relationship to the underlying $sl(N)$ loop algebra for $N \geqslant 3$ as the original Onsager's algebra to the $sl(2)$ loop algebra. The crucial property of Dolan–Grady relations—the fact that they generate an infinite series of integrals of motion in involution—is naturally present in this generalization.

A number of further questions present themselves. The first two of these are: What is the analog of Onsager's algebra for any Kac–Moody Lie algebra, and what are the corresponding analogs of the Dolan–Grady conditions? The answers are quite straightforward and we intend to report on this in a forthcoming publication.

Another problem concerns the representation theory of the $sl(N)$ Onsager's algebra. The classification of finite-dimensional representations

of $\mathscr{A}_N$ should be carried out in order to obtain an analog of the eigenvalue formula (9). Finally, we need to find examples of models with Hamiltonians of the form (10) that satisfy the conditions (11), (12) and cannot be mapped onto free fermions. Some special cases of the spin chains associated with the $sl(N)$ chiral Potts models[22] seem to be natural candidates for such models.

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